General Topology and its Applications 5 (1975) 181–194

$\mathcal{G}$-SOUSLIN DIAGONALS AND A THEOREM OF LORCH AND TONG

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Received 9 October 1973

In the first part of this paper it is proved that a property of topological spaces recently formulated by Lorch and Tong is equivalent to G_δ diagonality, thereby yielding an alternate proof of a recent result of theirs. This equivalence also yields several related theorems. In the second part it is shown that analogous results are valid for spaces with $\mathcal{G}$-Souslin diagonals. Consideration of these analogs yields the corollary that compact Hausdorff spaces which are countable unions of metrizable $\mathcal{G}$-Souslin subspaces are metrizable.

AMS Subj. Class.: 54E35, 54E45, 54H05, 02K30.

diagonal	metrizable
G_δ	Souslin (Suslin)

1. Introduction and notation

The subject of G_δ diagonals apparently originates with Šneĭder's theorem that compact Hausdorff spaces having G_δ diagonals are metrizable [11]. It is a subject which has occupied the attention of several contemporary authors (Burke [5], Ceder [6] and Okuyama [9]).

The first part of this paper (Section 2) continues the study of the G_δ diagonal property. In particular, it is proved that a property of topological spaces recently formulated by Lorch and Tong in [8] is equivalent to G_δ diagonality. This yields an alternative proof of the principal result of [8] (see Corollary 2.7). Several other equivalences and related theorems are also established, the proofs of which rely on the Lorch–Tong characterization. The notion of "α^{th}-order diagonal" (Definition 1.2) is introduced.

In a recent article [3], the author proved that every compact Hausdorff space with a *$\mathcal{G}$-Souslin* diagonal is metrizable. Analogs of the characterizations of G_δ diagonality, including those of Ceder and Lorch and Tong, are given for $\mathcal{G}$-Souslin diagonality in Theorem 3.3 and Remark 3.4. Some of these characterizations, as well as Theorem 3.9, are established by Theorem 3.7, which asserts that countable products of $\mathcal{G}$-Souslin subsets are $\mathcal{G}$-Souslin in the product space. It is a corollary to Theorem 3.9 that compact Hausdorff spaces which are countable unions of metrizable $\mathcal{G}$-Souslin subspaces are metrizable. This relates to a question posed by Alexandroff and Urysohn in [1].

1.1. Notation. Let T be a set and let $x \in X$. The element f of X^T defined by $f(t) = x$ for all $t \in T$ will be denoted x^T.

By S we shall denote the set of finite sequences of positive integers. Let N be the set of (strictly) positive integers with the discrete topology. The space $\mathbf{N}^{\mathbf{N}}$ will have the product topology. By $(t_1, t_2, ...) | n$ we mean $(t_1, t_2, ..., t_n)$. If $m \geq n$, then $(t_1, t_2, ..., t_m)| n$ denotes $(t_1, t_2, ..., t_n)$. For $i \in \mathbf{N}^{\mathbf{N}}$, i_n denotes $i(n)$.

The expression $s < i$ for $i \in \mathbf{N}^{\mathbf{N}}$ means $s = i | n$ for some $n \in \mathbf{N}$. By $s \leq (t_1, t_2, ..., t_m)$ we mean $s = (t_1, t_2, ..., t_n)$ for some $n \leq m$.

Let A be a set containing an element a. If $\{X_t : t \in A\}$ is a collection of topological spaces, then π_a denotes the projection map of $\Pi\{X_t : t \in A\}$ onto X_a.

Let X be a topological space. The family of open sets of X will be denoted $\mathcal{G}(X)$. A *$\mathcal{G}$-Souslin* set is one of the form

$$\bigcup_{s<i} \bigcap_{i \in N^n} H(s)$$

for some $H : S \to \mathcal{G}(X)$.

1.2. Definition. (i) A statement "let $\alpha \in [\gamma, \delta]$" will be understood to mean "let α be an ordinal number with $\gamma \leq \alpha \leq \delta$".

(ii) For every $\alpha \in [2, \omega]$, the *α^{th}-order diagonal* of X is the subset $\{x^\alpha : x \in X\}$ of X^α. The *diagonal of X* will refer to the second-order diagonal of X.

In the context of G_δ diagonals, it is natural to restrict the range of α as in Definition 1.2 because of the following theorem.

1.3. Theorem. *For any space X and any uncountable set A, the set $\{x^A : x \in X\}$ is not a G_δ (in fact, not $\mathcal{G}$-Souslin) in X^A.*

Proof. Assume, on the contrary, that

$$\{x^A : x \in X\} = \bigcup_{s<i} \bigcap_{i \in N^n} H(s),$$

with $U(s)$ open in X^A for every s in S. Let x be any element of X. Thus $x^A \in \bigcap \{U(s) : s < i\}$ for some i in $\mathbf{N}^{\mathbf{N}}$. For each n in $\mathbf{N}$, therefore, a finite family

$$\{V_a^n : a \in A_n \subseteq A\}$$

of sets open in X exists for which

$$x^A \in \bigcap \{\pi_a^{-1}(V_a^n) : a \in A_n\} \subseteq U(i \mid n)\,.$$

It follows that

$$(\#) \qquad x^A \in \bigcap \left\{\pi_a^{-1}(V_a^n) \colon (a, n) \in \bigcap_{m=1}^{\infty} (A_m \times \{m\})\right\} \subseteq \bigcap_{n=1}^{\infty} U(i \mid n) \subseteq \Delta.$$

Since $\bigcup_{n=1}^{\infty} A_n$ is countable, an element, b, of $A - \bigcup_{n=1}^{\infty} A_n$ exists. Let $y \in X$ with $y \neq x$. It follows from equation (#) that $f \in \Delta$, where f is defined by

$$f(a) = \begin{cases} x & \text{if } a \neq b, \\ y & \text{if } a = b. \end{cases}$$

This is absurd, and so Δ is not $\mathcal{G}$*-Souslin.* □

2. G_δ diagonals

The following well-known theorem is due to Šneĭder [11]. (Generalizations can be found, for example, in [4] and [9].)

2.1. Theorem. *A compact* T_2 *space is metrizable if and only if it has a* G_δ *diagonal.* □

2.2. Definition. We shall call a set *trivial* if it contains at most one element.

2.3. Definition. A space X has the *Lorch–Tong property* if, for each element x of X, there is a countable set $\{V(x, n) \colon n \in \mathbf{N}\}$ of open neighborhoods of x such that $\bigcap_{n=1}^{\infty} V(x_n, n)$ is trivial for every $\{x_n\}_{n=1}^{\infty} \subseteq X$.

2.4. Definition. We shall say that X has the *Ceder property* if there is a sequence $\mathcal{L}_n$ of open covers of X such that

$$\{x\} = \bigcap_{n=1}^{\infty} \operatorname{st}(x, \mathcal{L}_n)$$

for each x in X.

2.5. Theorem. *The following are equivalent for a topological space X.*

(1) *X has the Lorch–Tong property;*

(2) *X has the Ceder property;*

(3) *there is an ordinal number α in $[2, \omega]$ such that the α^{th}-order diagonal of X is a G_δ;*

(4) *X has a G_δ diagonal;*

(5) *for every α in $[2, \omega]$, the α^{th}-order diagonal of X is a G_δ.*

2.6. Remark. It is a consequence of Theorem 2.9 below (in which use is made of Theorem 2.5) that the following can be added to the list of equivalences in Theorem 2.5.

(6) *There is an ordinal number α in $[1, \omega]$ such that X^α has a G_δ diagonal.*

(7) *For every α in $[1, \omega]$, the space X^α has a G_δ diagonal.*

Proof of Theorem 2.5. It is obvious that (5) implies (4) which implies (3). That (2) and (4) are equivalent is proved in [6].

(3) *implies* (1). Let $\{x_n\}_{n=1}^{\infty} \subseteq X$. Suppose that $\alpha \in (1, \omega)$. Let $\{x^\alpha : x \in X\} = \bigcap_{n=1}^{\infty} G_n$, with G_n open in X^α for each n.

For each x in X and n in N, let $V(x, n)$ be an open neighborhood of x for which $V(x, n)^\alpha \subseteq G_n$. We have

$$\left[\bigcap_{n=1}^{\infty} V(x_n, n)\right]^\alpha = \bigcap_{n=1}^{\infty} [V(x_n, n)]^\alpha \subseteq \bigcap_{n=1}^{\infty} G_n = \{x^\alpha : x \in X\},$$

and so $\bigcap_{n=1}^{\infty} V(x_n, n)$ contains at most one point.

Consider now the case

$$\{x^\omega : x \in X\} = \bigcap_{n=1}^{\infty} G_n ,$$

where G_n is open in X^ω for each n.

For each x in X and n in $\mathbf{N}$, let $V(x, n)$ be an open neighborhood of x for which

$$\underbrace{V(x, n) \times V(x, n) \times \ldots \times V(x, n)}_{m_n \text{ factors}} \times X \times X \times \ldots \subseteq G_n$$

for some $m_n \in \mathbf{N}$.

It follows that $\bigcap_{n=1}^{\infty} V(x_n, n)$ contains at most one point.

(1) *implies* (5). Suppose that X satisfies the Lorch–Tong property with the system

$$\{U(x, n) : (x, n) \in X \times \mathbf{N}\}$$

of open neighborhoods as in the definition. Let α be finite, and define L by

$$L = \bigcap_{n=1}^{\infty} \bigcup_{x \in x} U(x, n)^{\alpha} .$$

We prove that $\{x^{\alpha} : x \in X\} = L$.

It is obvious that $\{x^{\alpha} : x \in X\} \subseteq L$. Let $(y_1, y_2, \ldots, y_{\alpha}) \in L$. Thus

$$(y_1, y_2, \ldots, y_{\alpha}) \in \bigcup_{x \in X} U(x, n)^{\alpha}$$

for every n in $\mathbf{N}$. A sequence $\{x_n\}_{n=1}^{\infty}$ exists, therefore, for which

$$(y_1, y_2, \ldots, y_{\alpha}) \in U(x_n, n)^{\alpha}$$

for every n in $\mathbf{N}$, and so

$$\{y_1, y_2, \ldots, y_{\alpha}\} \subseteq \bigcap_{n=1}^{\infty} U(x_n, n) .$$

Since $\bigcap_{n=1}^{\infty} U(x_n, n)$ contains at most one element, we have

$$(y_1, y_2, \ldots, y_{\alpha}) \in \{x^{\alpha} : x \in X\} .$$

In the case of X^{ω}, let

$$F = \bigcap_{n=1}^{\infty} \bigcap_{x \in X} \pi_1^{-1}(\widetilde{U}(x, n)) \times \pi_2^{-1}(\widetilde{U}(x, n)) \times \ldots \times \pi_n^{-1}(\widetilde{U}(x, n)),$$

where $\widetilde{U}(x, n) = U(x, 1) \cap U(x, 2) \cap \ldots \cap U(x, n)$. We prove that $\{x^{\omega} : x \in X\} = F$ in a manner similar to that above. $\square$

The following corollary is the principal result of [8]. Lorch and Tong remark on the existence of a proof using the Urysohn Metrization Theorem. The proof which they supply in [8] is for use on "subsequent occasions".

2.7. Corollary. *A compact Hausdorff space is metrizable if and only if it has the Lorch–Tong property. (In fact, any of the conditions* (1)–(5) *of Theorem* 2.5 *or conditions* (6) *or* (7) *of Remark* 2.6 *are necessary and sufficient for the metrizability of a compact Hausdorff space.)*

Proof. This is an immediate consequence of Theorem 2.1. □

2.8. Theorem. *Let* $\alpha \in [1, \omega]$. *For every ordinal* β *in* $[1, \alpha)$, *let* X_β *be a topological space containing a* G_δ *set* H_β. *Then* $\Pi\{H_\beta : \beta \in [1, \alpha)\}$ *is a* G_δ *in* $\Pi\{X_\beta : \beta \in [1, \alpha)\}$.

Proof. See [7, p. 346]. □

The proof of the following is left to the reader.

2.9. Theorem. *Let* α *be an ordinal number in* $[1, \omega]$, *and let* $\{Y_\beta : 1 \leqslant \beta < \alpha\}$ *be a collection of topological spaces. Then* $\Pi\{Y_\beta : 1 \leqslant \beta < \alpha\}$ *has a* G_δ *diagonal if and only if* Y_β *has a* G_δ *diagonal for every* $\beta \in [1, \alpha)$. □

3. $\mathcal{G}$-Souslin diagonals

The pertinence of $\mathcal{G}$-Souslin diagonals arises from the following theorem of the author's [3] and its corollary.

3.1. Theorem. *Every compact* $\mathcal{G}$*-Souslin set is a* G_δ.

Proof. If K is countably compact, where $K = \bigcup_{t \in \mathbb{N}^{\mathbb{N}}} \bigcap_{s < t} H(s)$ and $\mathcal{U} = \{H(s) : s \in S\}$ consists of open sets, then

$$K = \bigcap \{\bigcup \mathcal{M} : \mathcal{M} \subseteq \mathcal{U} \text{ is a finite cover of } K\}. \square$$

3.2. Corollary. *A compact Hausdorff space is metrizable if and only if it has a* $\mathcal{G}$*-Souslin diagonal.*

Proof. The corollary follows from Theorems 3.2 and 2.1. □

(1) and (2) of the following theorem are analogs of the Lorch–Tong and the Ceder characterizations of G_δ diagonality, respectively.

3.3. Theorem. *The following are equivalent.*

(1) *Every element x of X is contained in a $\mathcal{G}$-Souslin set*

$$\bigcup_{i\in \mathbf{N}^\mathbf{N}} \bigcap_{s<i} H(x, s)$$

such that for every i in $\mathbf{N}^\mathbf{N}$ and every $\{x_n\}_{n=1}^\infty \subseteq X$, $\bigcap_{n=1}^\infty H(x_n, i\,|\,n)$ is trivial.

(2) *For every s in S, there is a family $\mathcal{L}(s)$ of open sets such that*

(i) $$X = \bigcup_{i\in \mathbf{N}^\mathbf{N}} \bigcap_{s<i} L(s),$$

where $L(s)$ is defined as $\bigcup \mathcal{L}(s)$, and

(ii) *for every $i \in \mathbf{N}^\mathbf{N}$ and every $x \in \bigcap_{n=1}^\infty L(i\,|\,n)$, we have*

$$\{x\} = \bigcap_{n=1}^\infty \operatorname{st}(x, \mathcal{L}(i\,|\,n)).$$

(3) *There is an ordinal number α in [2, ω] such that the α^{th}-order diagonal of X is $\mathcal{G}$-Souslin.*

(4) *X has a $\mathcal{G}$-Souslin diagonal.*

(5) *For every α in [2, ω], the α^{th}-order diagonal of X is $\mathcal{G}$-Souslin.*

3.4. Remark. It is a consequence of Theorem 3.8 below that the following two equivalent properties may be added to those in Theorem 3.3.

(6) *There is an ordinal number $\alpha \in [1, \omega]$ such that X^α has a $\mathcal{G}$-Souslin diagonal.*

(7) *For every α in [1, ω], the space X^α has a $\mathcal{G}$-Souslin diagonal.*

Proof of Theorem 3.5. It is obvious that (5) implies (4), which implies (3).

(3) *implies* (1). Suppose that for some α in [2, ω],

$$\{x^\alpha : x \in X\} = \bigcup_{i\in \mathbf{N}^\mathbf{N}} \bigcap_{s<i} E(s),$$

where $E(s)$ is open in X^α for every s in S.

Assume first that $\alpha < \omega$. For each x in X and s in S, choose an open set $H(x, s)$ of X as follows.

If $x^\alpha \in E(s)$, then

$$x^\alpha \in [H(x, s)]^\alpha \subseteq E(s),$$

and if $x^\alpha \notin E(s)$, then $H(x, s) = \emptyset$. It follows that $x \in \bigcup_{i \in \mathbb{N}^\mathbb{N}} \bigcap_{s<i} H(x, s)$.

Now let $\{x_n\}_{n=1}^\infty \subseteq X$, and let $i \in \mathbb{N}^\mathbb{N}$; then $\bigcap_{n=1}^\infty H(x_n, i|n)$ can be seen to be trivial.

We now consider the case $\alpha = \omega$. For each (x, s) in $X \times S$, choose an open set $H(x, s)$ as follows. If $x^\omega \in E(s)$, then

$$x^\omega \in \bigcap_{m=1}^{n(s)} \pi_m^{-1}[H(x, s)] \subseteq E(s),$$

for some $n(s) \in \mathbb{N}$, and if $x^\omega \notin E(s)$, then $H(x, s) = \emptyset$.

It follows that $x \in \bigcup_{i \in \mathbb{N}^\mathbb{N}} \bigcap_{s<i} H(x, s)$.

The set $\bigcap_{n=1}^\infty H(x_n, i|n)$ can be seen to be trivial for every $\{x_n\}_{n=1}^\infty \subseteq X$.

(1) *implies* (5). We show first that for each α in $(1, \omega)$, $\{x^\alpha : x \in X\}$ is $\mathcal{G}$-Souslin in X^α.

For each s in S, define $V(s)$ by

$$V(s) = \bigcap_{x \in X} [H(x, s)]^\alpha .$$

It follows that $\{x^\alpha : x \in X\} = P$, where P is defined by

$$P = \bigcup_{i \in \mathbb{N}^\mathbb{N}} \bigcap_{s<i} V(s).$$

We now show that $\{x^\omega : x \in X\}$ is $\mathcal{G}$-Souslin in X^ω. For each (x, s) in $X \times S$, define $\widetilde{H}(x, s)$ by

$$\widetilde{H}(x, s) = \bigcap_{t \leq s} H(x, t) .$$

For every s in S, define $W(s)$ by

$$W(s) = \bigcup_{x \in X} \bigcap_{n=1}^{|s|} \pi_n^{-1}[\widetilde{H}(x, s)] .$$

It can be shown that $\{x^\omega: x \in X\} = T$, where T is defined by

$$T = \bigcup_{i \in \mathbf{N}^\mathbf{N}} \bigcap_{s<i} W(s)\,.$$

(4) *implies* (2). As before, let

$$\{x^2: x \in X\} = \bigcup_{i \in \mathbf{N}^\mathbf{N}} \bigcap_{t<i} E(t)\,,$$

and define $H : X \times S \to \mathcal{G}(X)$ as follows. If $x^2 \in E(s)$, then $H(x, s)$ is an open set for which

$$x^2 \in [H(x, s)]^2 \subseteq E(s)\,,$$

and if $x^2 \notin E(s)$, then $H(x, s) = \emptyset$.

For each s in S, define $\mathcal{L}(s)$ by

$$\mathcal{L}(s) = \{H(x, s): x \in X\}\,.$$

The following equation holds:

$$X = \bigcup_{i \in \mathbf{N}^\mathbf{N}} \bigcap_{s<i} L(s)\,,$$

where $L(s) = \bigcup \mathcal{L}(s)$. Now let $i \in \mathbf{N}^\mathbf{N}$ and let $x \in \bigcap_{n=1}^\infty L(i|n)$. It is obvious that

$$\{x\} \subseteq \bigcap_{n=1}^\infty \operatorname{st}(x, \mathcal{L}(i|n))\,.$$

If $z \neq x$ then $(x, z) \notin \Delta$, and so

$$(x, z) \notin \bigcap \{E(s): s < i\}\,.$$

Thus $(x, z) \notin E(i|n)$ for some n in $\mathbf{N}$, and so $(x, z) \notin [H(y, i|n)]^2$ for every y in X. In particular, $z \notin H(y, i|n)$ for all y in X, from which it follows that

$$z \notin \operatorname{st}(x, \mathcal{L}(i|n)) \supseteq \bigcap_{m=1}^\infty \operatorname{st}(x, \mathcal{L}(i|m))\,.$$

We also have

$$\{x\} = \bigcap_{n=1}^\infty \operatorname{st}(x, \mathcal{L}(i|n))\,.$$

(2) *implies* (4). For every $s \in S$, let

$$H(s) = \bigcup \{U^2 : U \in \mathcal{L}(s)\}.$$

It can be shown that

$$\{x^2 : x \in X\} = \bigcup_{f \in \mathbb{N}^\mathbb{N}} \bigcap_{t < f} H(t). \square$$

3.5. Remark. Since (1) of the previous theorem implies that the singletons of X are G_δ's, it follows that every space with a $\mathcal{G}$-Souslin (e.g. G_δ) diagonal has G_δ singletons. The converse is false. In fact, even compact Hausdorff first countable spaces do not necessarily have $\mathcal{G}$-Souslin diagonals, otherwise they would necessarily be metrizable by Theorem 2.1. The lexicographic order topology on the unit square is a counterexample.

3.6. Corollary. *A necessary and sufficient condition for a compact Hausdorff space to be metrizable is that it satisfy one of the conditions* (1), (2), (3), (4) *or* (5) *of Theorem* 3.3, *or* (6) *or* (7) *mentioned in Remark* 3.4.

Proof. This is an immediate consequence of Theorem 3.1 and 2.1.$\square$

3.7. Theorem. *Let $\alpha \in [2, \omega]$. For every ordinal β in $[1, \alpha)$, let X_β be a topological space containing a $\mathcal{G}$-Souslin set H_β. Then $\Pi\{H_\beta : \beta \in [1, \alpha)\}$ is $\mathcal{G}$-Souslin in $\Pi\{X_\beta : \beta \in [1, \alpha)\}$.*

Proof. Let

$$H_\beta = \bigcup_{f \in \mathbb{N}^\mathbb{N}} \bigcap_{s < f} U(s, \beta),$$

with $U(s, \beta)$ open in X_β for every β in $[1, \alpha)$.

Assume first that $\alpha < \omega$. We define $V : S \to \mathcal{G}(\Pi_{\beta=1}^{\alpha-1} X_\beta)$ as follows. If $n < \alpha - 1$, then we set

$$V(t_1, t_2, \ldots, t_n) = U(t_1) \times U(t_2) \times \ldots \times U(t_n) \times X_{n+1} \times X_{n+2} \times \ldots \times X_{\alpha-1}.$$

If $n = k(\alpha - 1) + r$, where $k \geqslant 1$ and $0 < r < \alpha - 1$ $(k, r \in \mathbb{N})$, then we put

$$
\begin{aligned}
V(t_1, t_2, \ldots, t_n) = U(t_1, t_\alpha, t_{2(\alpha-1)+1}, \ldots, t_{k(\alpha-1)+1})\\
\times U(t_2, t_{(\alpha-1)+2}, t_{2(\alpha-1)+2}, \ldots, t_{k(\alpha-1)+2})\\
\times \ldots \times U(t_r, t_{(\alpha-1)+r}, \ldots, t_{k(\alpha-1)+r})\\
\times U(t_{r+1}, t_{(\alpha-1)+r+1}, \ldots, t_{(k-1)(\alpha-1)+r+1})\\
\times U(t_{r+2}, t_{(\alpha-1)+r+2}, \ldots, t_{(k-1)(\alpha-1)+r+2})\\
\times \ldots \times U(t_{\alpha-1}, t_{2(\alpha-1)}, \ldots, t_{k(\alpha-1)}) .
\end{aligned}
$$

Finally, if $n = k(\alpha - 1)$, where $k \geqslant 1$, then we set

$$
\begin{aligned}
V(t_1, t_2, \ldots, t_n) = U(t_1, t_{(\alpha-1)+1}, \ldots, t_{(k-1)(\alpha-1)+1})\\
\times U(t_2, t_{(\alpha-1)+2}, \ldots, t_{(k-1)(\alpha-1)+2})\\
\times \ldots \times U(t_{\alpha-1}, t_{2(\alpha-1)}, \ldots, t_{k(\alpha-1)}) .
\end{aligned}
$$

We show that

$$\prod_{\beta=1}^{\alpha-1} H_\beta = \bigcup_{i\in \mathbf{N}^{\mathbf{N}}} \bigcap_{s<i} V(s) .$$

If $(x_1, x_2, \ldots, x_{\alpha-1}) \in \Pi_{\beta=1}^{\alpha-1} H_\beta$, then for each $1 \leqslant p < \alpha$,

$$x_p \in \bigcap_{s<i(p)} \{U(s, p)\}$$

for some $i(p)$ in $\mathbf{N}^{\mathbf{N}}$. Thus

$$(x_1, x_2, \ldots, x_{\alpha-1}) \in V(j \mid n) \text{ for all } n \text{ in } \mathbf{N},$$

where

$$j = (i(1)_1, i(2)_1, \ldots, i(\alpha-1)_1, i(1)_2, i(2)_2, \ldots, i(\alpha-1)_2, \ldots) .$$

(By $i(k)_l$ we mean the l^{th} entry in $i(k)$.) It follows that

$$(x_1, x_2, \ldots, x_{\alpha-1}) \in \bigcup_{i\in \mathbf{N}^{\mathbf{N}}} \bigcap_{s<i} V(s) .$$

The converse inclusion can be shown without difficulty.

We now turn to the case $\alpha = \omega$, defining

$$W : S \to \mathcal{G}\left(\prod_{\beta=1}^{\infty} X_\beta\right)$$

as follows:

$$W(t_1) = U(t_1) \times X_2 \times X_3 \times \ldots$$

$$W(t_1, t_2) = U(t_1, t_2) \times X_2 \times X_3 \times \ldots$$

$$W(t_1, t_2, t_3) = U(t_1, t_2) \times U(t_3) \times X_3 \times X_4 \times \ldots$$

$$W(t_1, t_2, t_3, t_4) = U(t_1, t_2, t_4) \times U(t_3) \times X_3 \times X_4 \times \ldots$$

$$W(t_1, t_2, \ldots, t_5) = U(t_1, t_2, t_4) \times U(t_3, t_5) \times X_3 \times X_4 \times \ldots$$

$$W(t_1, t_2, \ldots, t_6) = U(t_1, t_2, t_4) \times U(t_3, t_5) \times U(t_6) \times X_4 \times X_5 \times \ldots$$

$$W(t_1, t_2, \ldots, t_7) = U(t_1, t_2, t_4) \times U(t_3, t_5) \times U(t_6) \times U(t_7) \times X_5 \times X_6 \times \ldots$$

$$W(t_1, t_2, \ldots, t_8) = U(t_1, t_2, t_4, t_8) \times U(t_3, t_5) \times U(t_6) \times U(t_7) \times X_5 \times X_6 \times \ldots$$

$$\vdots$$

It follows, in a manner similar to the first part of the proof, that

$$\prod_{\beta=1}^{\infty} H_\beta = \bigcup_{t \in \mathbf{N}^{\mathbf{N}}} \bigcap_{s < t} W(s) . \square$$

3.8. **Theorem.** *Let α be an ordinal number in $[1, \omega]$, and let $\{Y_\beta : 1 \leqslant \beta < \alpha\}$ be a collection of topological spaces. Then $\Pi\{Y_\beta : 1 \leqslant \beta < \alpha\}$ has a $\mathcal{G}$-Souslin diagonal if and only if Y_β has a $\mathcal{G}$-Souslin diagonal for every β in $[1, \alpha)$.*

Proof. The proof of the sufficiency part of this theorem is almost identical to that of Theorem 2.9.

To prove the necessity part we show that if $X \times Y$ has a $\mathcal{G}$-Souslin diagonal, then so does X.

Let

$$\{U(z, s) : (z, s) \in (X \times Y) \times S\}$$

be the system of open sets defined by Theorem 3.3 (1).

Let y_0 be an element of Y, and define $V : X \times S \to \mathcal{G}(X)$ by

$$V(x, s) = \{v \in X : (v, y_0) \in U((x, y_0), s)\}.$$

Since

$$(x, y_0) \in \bigcup_{i \in \mathbf{N}^\mathbf{N}} \bigcap_{s < i} U((x, y_0), s),$$

it follows that

$$x \in \bigcup_{i \in \mathbf{N}^\mathbf{N}} \bigcap_{s < i} V(x, s).$$

Suppose that $\{x_n\}_{n=1}^\infty \subseteq X$. We have, for every i in $\mathbf{N}^\mathbf{N}$,

$$[\bigcap_{n=1}^\infty V(x_n, i|n)] \times \{y_0\} = \bigcap_{n=1}^\infty [V(x_n, i|n) \times \{y_0\}] \subseteq \bigcap_{n=1}^\infty U((x_n, y_0), i|n),$$

which is trivial. Thus $\bigcap_{n=1}^\infty V(x_n, i|n)$ is trivial, and so X has a $\mathcal{G}$-Souslin diagonal by Theorem 3.3 (1). □

Smirnov [10] proved that no compact non-metrizable Hausdorff space is the countable union of *second countable* metrizable subspaces. In the same vein, we have the following theorem.

3.9. Theorem. *No compact non-metrizable Hausdorff space is the countable union of $\mathcal{G}$-Souslin metrizable subspaces.*

(More generally, a topological space has a $\mathcal{G}$-Souslin diagonal if and only if it is a countable union of $\mathcal{G}$-Souslin subspaces each of which has a $\mathcal{G}$-Souslin diagonal.)

3.10. Remark. (1) That the parenthetical statement is indeed more general is a consequence of Corollary 3.2.

(2) As Alexandroff and Urysohn pointed out in [1], it is easy to see that compact Hausdorff spaces are metrizable if they are the unions of countably many *open* metrizable subspaces.

3.11. Lemma. *If A is a $\mathcal{G}$-Souslin subspace of X, and B is $\mathcal{G}$-Souslin in A, then B is $\mathcal{G}$-Souslin in X.*

Proof. Let A be $\mathcal{G}$-Souslin in X and let $B = \bigcup_{t \in \mathbb{N}^{\mathbb{N}}} \bigcap_{s<t} L(s)$, with $L : S \to \mathcal{G}(A)$. For each s in S, $L(s) = A \cap V(s)$ for some $V(s)$ in $\mathcal{G}(X)$. Thus

$$B = \bigcup_{t \in \mathbb{N}^{\mathbb{N}}} \bigcap_{s<t} (A \cap V(s)) = A \cap \bigcup_{t \in \mathbb{N}^{\mathbb{N}}} \bigcap_{s<t} V(s) ,$$

the intersection of two $\mathcal{G}$-Souslin subsets of X, and so B is $\mathcal{G}$-Souslin in X. □

Proof of Theorem 3.9. Assume that $X = \bigcup_{m=1}^{\infty} H_m$, where each H_m is $\mathcal{G}$-Souslin. Let Δ be the diagonal in X^2, and Δ_m that in $H(m)^2$. Since $H(m)^2$ is $\mathcal{G}$-Souslin in X^2 by Theorem 3.7, and Δ_m is $\mathcal{G}$-Souslin in $H(m)^2$, it follows from Lemma 3.11 that Δ_m is $\mathcal{G}$-Souslin in X^2. Since $\Delta = \bigcup_{m=1}^{\infty} \Delta_m$ and $\mathcal{G}$-Souslin sets are closed under countable unions, the set Δ is $\mathcal{G}$-Souslin in X^2, a contradiction. □

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